Fluid Simulation using Shallow Water Equation

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ABSTRACT
In this paper, we present a technique, which shows how waves once generated from a small drop continue to ripple. Waves interact with each other and on collision change the form and direction. Once the waves strike the boundary, they return with the same speed and in sometime, depending on the delay, you can see continuous ripples in the surface. We use shallow water equation to achieve the desired output.

Author Keywords
Naïve Stroke Equation; Shallow Water Equation; Fluid Simulation; WebGL; Quadratic function.

INTRODUCTION
In early years of fluid simulation, procedural surface generation was used to represent waves as presented by authors in [12], [13] and [14]. There has been a significant progress in field of fluid simulation over the past few years. The improvements can be seen particularly with respect to application with various scenarios such as different material interaction, phase change and multiphase, visual accuracy. [1] The main difficulty encountered faced by researchers in this area is the collaboration of various factors on which fluid simulation depends. Gravity, velocity, viscosity, collision, wind, direction, waves, interaction with object and many more of these factors contribute to real life simulation of liquid [10]. Incorporating all the factors to represent real life experience is a challenge.

We make use of shallow water equation, which is derived from Naïve Stroke Equation [11]. We used two different applications and collaborated them together into one (Flat Wave plus Terrain). Shallow Water equation is a system of hyperbolic/Parabolic Partial Differential Equation. They govern water flow in oceans, coastal regions, estuaries, rivers and channel. As the name suggest, the main characteristic of shallow Water flows is that the vertical dimension is much smaller as compared to the horizontal dimension. Naïve Stroke Equation defines the motion of fluids and from these equations we derive SWE.

In Next section we talk about the related works under the heading Literature Review. Then we will explain the framework and other concepts necessary to understand the SWE and wave equation. In the section followed by it, we show the results achieved using our implementation. Then finally we talk about conclusion and future work.

LITERATURE REVIEW
In [2] author in detail explains the Shallow water equation with its derivation. Since then a lot of authors has used SWE to present the formation of waves in water. Authors in [3] present the visual simulation of water using SWE. The simulated output involved dry-bed zones and non-trivial bottom topographic. These are some real challenges to robustness and accuracy of discretization. Authors in [1] present a new method, which aims to enhance the shallow water simulation by the effect of overturning waves. They showcase a technique, which makes it possible to simulate scenes such as waves near a beach, and surf riding characters in real time. Researchers have experimented with not only even but also uneven surface as well. In [4], authors use Saint-Venant system for shallow water flows with non-flat bottom. The goal of the authors in [5] is to perform simulations that capture fluid effects from small drops up to the propagation of large waves. To achieve the desired result, the authors present a hybrid Simulation method, which combines 2D shallow water simulation with a full 3D free surface fluid simulation. The computation of shallow water equations in one dimensional with topography by Finite Volume methods is studied by the authors in [6]. In [7], authors have developed a simple scheme for treatment of vertical bed topography in shallow water flows. The shallow water equations including local energy loss terms are used to model the effect of vertical step on flows. SWE (non-flat bottom) have steady state solutions. In these the flux gradient are not zero but are exactly balanced by the source term. The challenge to design genuinely high order accurate numerical schemes in presented and solved by the authors in [8] which preserve exactly these steady state solutions. In this paper [9], a

moving boundary technique is developed to investigate wave run-up and rundown with the help of depth-integrated equations.

We can see a lot of research has been done in field of fluid mechanics that makes use of Shallow Wave Equations directly or after modifying few parameters. In this paper we also make use of SWE and try to show how waves behave when they strike with each other or with boundaries. In the framework we will first explain the Naïve Stroke Equation and then show how SWE is derived from it. We

FRAMEWORK

Before directly moving on to how we used the equation in our framework, we will explain the derivation of Shallow Wave Equations from Naïve Stroke Equations.

There are four basic steps required. Firstly, derive the Naïve Stroke equation itself from the conservation laws. Secondly, get an average of Naïve Stroke Equations to account for the turbulent nature of ocean flow. Thirdly and most importantly, we specify the boundary conditions for NSE. Then finally integrate NSE over depth.

Suppose we have applied the Conservation Laws and we receive the Naïve Stroke Equations given below.

- Naïve Stroke Equations:

  $\partial u/ \partial x + \partial v/\partial y + \partial w/\partial z = 0$ (1)

  $\partial(\rho u)/\partial t + \partial(\rho u^2)/\partial x + \partial(\rho uv)/\partial y + \partial(\rho uw)/\partial z = \partial(\tau_{xx}-\rho)/\partial x + \partial\tau_{xy}/\partial y + \partial\tau_{xz}/\partial z$ (2)

  $\partial(\rho v)/\partial t + \partial(\rho uv)/\partial x + \partial(\rho v^2)/\partial y + \partial(\rho vw)/\partial z = \partial\tau_{xy}/\partial x + \partial(\tau_{yy}-\rho)/\partial y + \partial\tau_{yz}/\partial z$ (3)

  $\partial(\rho w)/\partial t + \partial(\rho uw)/\partial x + \partial(\rho vw)/\partial y + \partial(\rho w^2)/\partial z = -\rho g + \partial\tau_{xz}/\partial x + \partial\tau_{yz}/\partial y + \partial(\tau_{zz}-\rho)/\partial z$ (4)

  where,

  ρ is the fluid density (kg/m$^{3)}$,

  g is acceleration due to gravity (m/s^2),

  v= (u v w) is the fluid velocity (m/s),

- Using Boundary Conditions:

  At the bottom (z=-b)

  - No slip u=v=0
  - No normal flow: $u\ \partial b/ \partial x + v\ \partial b/ \partial y + w=0$ (5)
  - Bottom Shear Stress:

    $\tau_{bx=}\ \tau_{xx}\ \partial b/ \partial x + \tau_{xy}\ \partial b/ \partial y + \tau_{xz}$ (6)

    Where, T_{bx} is specified bottom friction.

  At the free surface (z= ζ)

  - No relative Normal Flow:

    $\partial\zeta/ \partial t + u\ \partial\zeta/\partial x + v\ \partial\zeta/\partial y - w= 0$ (7)

  - ρ=0, done in (2)
  - Surface Sheer Stress:

    $T_{sx=}\ -\tau_{xx}\ \partial\zeta/ \partial x - \tau_{xy}\ \partial\zeta/ \partial y + \tau_{xz}$ (8)

Before we integrate over depth, we can examine the momentum equation for vertical velocity. By a scaling argument, all the terms except the pressure derivative and the gravity term are small. Then z-momentum equation collapse to:

$$\partial p/\partial z = \rho g$$

$$p= \rho g\ (\zeta -z)$$

This is the hydrostatic pressure distribution in equation 9.

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \zeta}{\partial x} \qquad (9)$$

We now integrate the continuity equation ∇ . V=0 from z=-b to z= ζ. Since both are dependent on t, x, y we use Leibniz Integral rule.

$$\begin{aligned} 0 &= \int_{-b}^{\zeta} \nabla \cdot \mathbf{v}\, dz \\ &= \int_{-b}^{\zeta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz + w|_{z=\zeta} - w|_{z=-b} \\ &= \frac{\partial}{\partial x}\int_{-b}^{\zeta} u\, dz + \frac{\partial}{\partial y}\int_{-b}^{\zeta} v\, dz - \left(u|_{z=\zeta}\frac{\partial \zeta}{\partial x} + u|_{z=-b}\frac{\partial b}{\partial x}\right) \\ &\quad - \left(v|_{z=\zeta}\frac{\partial \zeta}{\partial y} + v|_{z=-b}\frac{\partial b}{\partial y}\right) + w|_{z=\zeta} - w|_{z=-b} \end{aligned}$$

Defining depth average velocities as

$$\bar{u} = \frac{1}{H}\int_{-b}^{\zeta} u\, dz, \qquad \bar{v} = \frac{1}{H}\int_{-b}^{\zeta} v\, dz,$$

We can use our BC to get rid of the boundary terms. So the depth average continuity equation is:

$$\boxed{\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = 0} \quad (10)$$

Now if we integrate the LHS of x momentum equation over depth

$$\int_{-b}^{\zeta}[\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u^2 + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)]\,dz$$
$$= \frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H\bar{u}^2) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) + \begin{Bmatrix}\text{Diff. adv.}\\ \text{terms}\end{Bmatrix}$$

Integrating over depth gives us

$$\begin{cases} -\rho g H\frac{\partial \zeta}{\partial x} + \tau_{sx} - \tau_{bx} + \frac{\partial}{\partial x}\int_{-b}^{\zeta}\tau_{xx} + \frac{\partial}{\partial y}\int_{-b}^{\zeta}\tau_{xy} \\ -\rho g H\frac{\partial \zeta}{\partial y} + \tau_{sy} - \tau_{by} + \frac{\partial}{\partial x}\int_{-b}^{\zeta}\tau_{xy} + \frac{\partial}{\partial y}\int_{-b}^{\zeta}\tau_{yy} \end{cases}$$

Combining the depth integrated continuity equation with the LHS and RHS of the depth integrated x- and y-momentum equations, 2D SWE in conservative form are:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = 0$$
$$\frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H\bar{u}^2) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) = -gH\frac{\partial \zeta}{\partial x} + \frac{1}{\rho}[\tau_{sx} - \tau_{bx} + F_x]$$
$$\frac{\partial}{\partial t}(H\bar{v}) + \frac{\partial}{\partial x}(H\bar{u}\bar{v}) + \frac{\partial}{\partial y}(H\bar{v}^2) = -gH\frac{\partial \zeta}{\partial y} + \frac{1}{\rho}[\tau_{sy} - \tau_{by} + F_y]$$

These are the final Shallow Water Equations used in our code to generate the waves.

RESULT

As soon as you start the simulation of the code on a browser, Firefox or Chrome, you will see a drop of liquid falling into the square Figure 1.

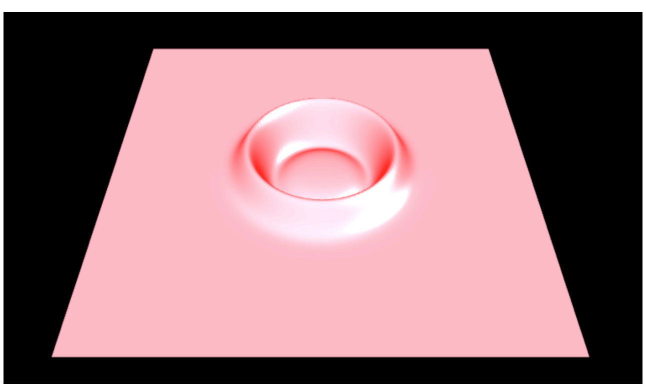

Figure 1 A Small drop of liquid when falls into the square pond in middle (X=0, Y=0).

Just after the drop falls, it forms a wave in the square liquid surface as show in figure 2.

Figure 2 The drop spreads and form waves.

The waves after colliding with the boundary come back and interact with other waves Figure 3.

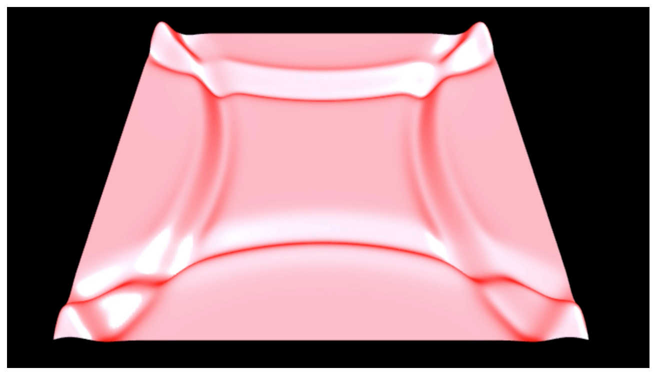

Figure 3 The waves collide with the boundary and return back.

Finally, after multiple collisions with boundary and other waves we get continuous waves which look like ripples Figure 4.

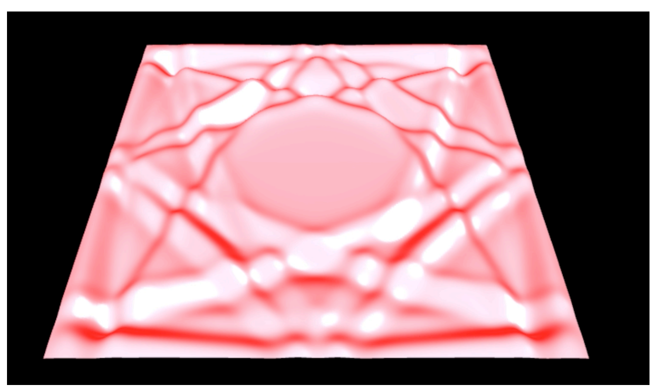

Figure 4 After multiple collisions with the boundary and other waves.

Figure 5 shows that the location of the drop can be modified and therefore the waves are formed accordingly.

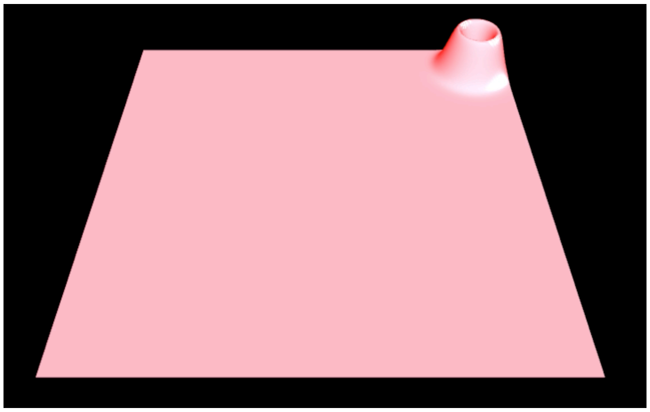

Figure 5 Drop of liquid at right-top corner.

CONCLUSION

Our work is mainly focused to help beginners who are new to the field of fluid simulation and provide them some insight in some techniques, which one can use to simulate waves. Shallow Water Equation is known to be one of the very famous techniques, which can be used to visualize the simulation of waves in water. We try to use it and present a view of how water waves behave in real time within a confined boundary. Every one of us has at one point or another have seen waves and tides in oceans and how they collide with one another to generate new form and direction. We make use of wave equation but we only considered the interaction between water particles. We didn't consider how water interacts with waterbed or sand; we neither considered other factors like viscosity of different materials and transparency of liquid. We plan on including more real time factors in our future project and present a more real time rendering of water waves. We would also like to show the behavior of waves when it comes in contact with other rigid objects like rocks.

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